EXTERIOR BALLISTICS OF BOWS AND ARROWS

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Exterior Ballistics of Bows and Arrows deals with the flight of an arrow after it leaves the bow. This article is concerned primarily with the maximum range that can be obtained with arrows of various designs, and with various initial velocities. When an arrow is shot from a bow, the distance that it will fly depends upon the following:

1 – The velocity with which it leaves the bow, called the "Initial Velocity";

2 – The angle of departure;

3 – The weight of the arrow;

4 – The resistance or friction or drag of the arrow in air;

5 – Velocity and direction of wind.

The velocity with which an arrow leaves the bow depends upon the force of the bow, the efficiency of the bow, and the mass of the arrow. The article will not deal with the relationship between these factors, which is called "Interior Ballistics", but will simply assume a given initial velocity when calculating the effect of arrow design on maximum range.

The angle of departure for maximum range is approximately 45 degrees with the horizontal. The actual angle of departure for maximum range will be calculated below.

Since the effect of wind on the maximum range is variable and hard to determine, no attempt will be made to evaluate its effects.

The resistance to motion of an arrow during its flight thru the air considerably reduces its range. Since this resistance depends upon the design of an arrow, the first step in determining the effect of arrow design on maximum range will be to determine its resistance.

Resistance varies with velocity. The higher the velocity the greater the resistance. It has been found in wind tunnel tests, and by the firing of a large number of projectiles, that the resistance varies as the square of the velocity for velocities below 800 feet per second .

In archery, we deal with velocities well below the value, and a simple equation for resistance can be written as follows:

$$R_f = KV^2 \qquad (1)$$

Where R_f = the resistance in pounds
K = a coefficient depending upon arrow design.
V = velocity in feet per second.

The coefficient K applies to only one design of arrow. If the value of K is measured for a given arrow, any change in design, such as a change in length, diameter, or size of feather, would require a new wind tunnel test to determine the new value of "K". To obviate an infinite number of wind tunnel tests, the separate factors in arrow design affecting the resistance coefficient can be evaluated, making it possible to calculate "K" for any design of arrow.

The resistance of arrow is made up of the following:

1 – The head-on resistance, which includes the rear end drag. This resistance depends upon the head-on area of the arrow, and will therefore vary as the square of the diameter. The head-on resistance will also vary in accordance with the shape of the head of the arrow, an ogival shaped head giving considerably less resistance than a blunt head.

2 – Skin friction of the arrow shaft. This varies with the surface area of the shaft and therefore varies with the length and diameter of the shaft.

3 – Skin friction of the feathers. This will vary with the total area of both sides of the feathers. The above can be expressed in terms of an equation as follows:

K= K'BD^2 + K" LD + K'''F (2)
Where K, K', K'' and K''' are coefficients.
B is a coefficient of form for the head of the arrow
D is the diameter of the arrow in inches.
L is the length of the arrow in inches.
F is the area of both sides of the feathers in square inches.

By determining the coefficients K, K', K'' and K''' in equation (2), the value of K for any design of arrow can be calculated. One way to determine these values accurately is by wind tunnel tests.

The author knows of only one wind tunnel experiment on arrows. The May-June, 1935, issue of Army Ordinance contained in an article entitled "the Bow as a Missile Weapon" by Vice Admiral W.L. Rodgers. He gives the results of the test made by Rear Admiral Moffet, chief of the Bureau of Aviation, on an arrow with ogival head, 26 inches long, 5/16 inch diameter, with 3 feathers 2 1/2 inches long, having a total feather area of 7.5 square inches.

At 200 feet per second(FPS), the resistance of this arrow was 0.039 pounds. Removing the feathers, the resistance was 0.016 pounds. These two tests make it possible to determine the value of K''' in formula (2) as follows;

0.039 - 0.016 - 0.023 pounds = the resistance of the 7 1/2 square inches of feathers at 200 FPS velocity.

Therefore K''' = 0.023/(7.5 X 2002) = 0.000000077

Since this arrow without feathers gave a resistance of 0.016 pounds, another equation can be written as follows;

$$(K'BD^2 + K''LD) = 0.016 \qquad (3)$$

However this equation has three unknowns, K', K'', and B. By determining any two of these, the third can be calculated from the equation.

Wind tunnel tests on various shaped objects give a clue to the head-on resistance represented by the factor K'BD2. For example, K'B for a cone varies from 0.0000033 for a 60 degree included angle to 0.0000015 for a 20 degree included angle. The cone on most parallel piles used on arrows has about a 60 degree included angle. However, the above values are based on a cone with a flat base, whereas arrows are usually tapered towards the rear, and the ends are slightly rounded. This should reduce the resistance coefficient somewhat, but in an arrow this is largely offset by the actual nocks, which create a disturbance in the streamline flow. This value of K'B = 0.000003 for an arrow with a parallel pile is probably a fair approximation.

For a bullet shaped or ogival point, a still smaller value can be expected. The best known formula for the resistance of a bullet is that of Mayeski, which for velocities below 800 FPS gives K'B = 0.00000135, and where B, the coefficient of form, equals 1, for the ogival head.

When used on an arrow, this coefficient can again be slightly reduced because of slightly improved rear end conditions as follows;

K'B = 0.0000013
If B = 1 for ogivals heads and
B = 2.3 for parallel piles, the value of K' becomes 0.0000013.

The value of B for a blunt head is approximately 7.0. Having established values for K' and B, these values can be substituted in equation (3) as follows:

$$0.016 = (0.0000013 \times 1 \times 0.312^2 + K'' \times .312 \times 26) \times 200^2$$ and solving for K''
$K'' = 0.000000035$

Having established values for K', K'', K''', and B, equation (2) can be written as follows:

$$K = 0.0000013BD^2 + 0.000000035LD + 0.000000077F \qquad (4)$$
B = 1 for ogival or bullet shaped heads
B = 2.3 for parallel pile heads
B = 7.0 for blunt heads
D = Maximum diameter of arrow in inches
L = Length of shaft in inches
F = Area of both sides of feather's in square inches and equation (1) becomes:

$$Rf = (0.000001.3BD^2 + 0.000000035LD + 0.000000077F)V^2 \qquad (5)$$

Inspection of the above formula will indicate that the feathers are responsible for the large part of drag.

Formula (5) also indicates the advantage of celluloid vanes as compared to feathers. Both the resistance of the body of the arrow and of the feathers depend upon their respective surface areas. Therefore, , the coefficients for these two factors as determined by the wind tunnel tests show the difference in resistance between a smooth surface, such the body of an arrow, and a rough surface, such as a feather. The coefficient for the surface of the feather was found to be more than double the coefficient for the surface of the arrow. By using a smooth celluloid vane with a surface similar to the body surface of an arrow, the feather resistance can be reduced by more than 50%.

In order to check formula (5), we can make use of a method commonly used in artillery for measuring the resistance of projectiles. This consists of measuring the velocity of the projectile at points some distance apart, either by chronograph or a ballistic pendulum. The formula for resistance then becomes:

$$R_f = (W/2gx)(V_1^2 - V_2^2) \qquad (6)$$

Where W = weight of the projectile in pounds
g = Acceleration of gravity - 32.2 ft/s^2
x = Distance in feet between points of measurement of V1 and V2.
V1 = Velocity in feet per second at the initial point of measurement
V2 = Velocity in feet per second at distance x from first point

English, in his article on "Exterior Ballistics of Arrows" in December 1930 issue of the Journal of the Franklin Institute, gives the results of tests he made with a ballistic pendulum on two weights of arrows with different initial velocities. He determined the initial velocity at the bow and the striking velocity fifty yards from the bow for the first arrow, and forty yards from the bow for the second arrow.

For the first test he used an arrow weighing 0.0615 lbs. this arrow had an initial velocity of 155.4 feet per second and a striking velocity of 141.8 feet per second at a distance of 150 feet from the bow.

From formula (6)

$$R_f = (0.0615/64.4 \times 150) \times (155.4^2 - 141.8^2) = 0.0257 \text{ lb}$$

In order to compare this value with formula (5), we have the length of the arrow given as 31 inches and the diameter as 0.355 inches. English also gave the areas of three different size feathers used in his various tests, but unfortunately failed to designate which size was used on any given test. Assuming that he used feathers with an area of 1.11 inches per feather, then from formula (4) -

$$K = 0.0000013 \times 0.355^2 + 0.000000035 \times 0.355 \times 31 + 0.0000000767 \times 1.11 \times 6 = 0.00000106 \text{ and } R_f = 0.00000106V^2$$

The average velocity for the first test was 148.6 feet per second. Therefore:

$$R_f = 0.00000106(148.6)^2 = 0.0236 \text{ pounds}$$

The value of resistance as determined by actual test was accordingly 9% greater than the resistance as computed from formula (5).

Similarly, the second test by English on an arrow weighing 0.0851 pounds with an initial velocity of 138.5 feet per second and a striking velocity of 131.4 feet per second, 120 feet away, gives a value of:

$Rf = 0.0210$ pounds

Formula (5) for this arrow gives;

$Rf = 0.0190$ pounds

These two values also check within 10%. The discrepancies are either due to an error in the coefficients formula (4) or an error in the measurement of velocities by English. Because of the absence of sufficient wind tunnel tests on arrows, an error in the coefficients is entirely possible. However, since the discrepancy would only be 4% in the maximum range of the arrow, the check between resistance as calculated by formula (4) and as determined by English is remarkably close.

Therefore formula (5), with coefficients based on wind tunnel tests, probably the most accurate formula for resistance of an arrow that is available at the present time. Additional ballistic pendulum or wind tunnel tests will undoubtedly modify the coefficients to some extent, but such modification will be small and can have very little effect on the relative results obtained by its use in this article.

Having determined the resistance Rf of an arrow, the next step is to determine the ballistic coefficient Co. This is defined as:

$$Co = AV^2/Ra \qquad (7)$$

Where A = 0.00004676 for velocities less than 800 feet per second
V = velocity in feet per second
Ra = deceleration of the arrow due to air resistance

But $Ra = Rf/M = Rfg/W$ (from the formula: force=mass X acceleration

Then $Co = AV^2W/Rfg$

But $Rf = KV^2$ from (1)

Then $Co = AW/Kg$

$$\text{Or } Co = 0.0000000002075W/K \qquad (8)$$

Where W = weight of the arrow in grains and K is obtained from formula (4)

The ballistic coefficient Co is therefore dependent upon permanent features of a given arrow such as size, weight, and feather area, and can be readily computed. The coefficient Co may be thought of as measuring the "ranging power" of the arrow. It varies directly with the weight and inversely with the coefficient of resistance K.

It should be noted here that the value of Co is based on standard atmospheric conditions corresponding to 59 degrees Fahrenheit and a barometric pressure of 29.53 and relative humidity of 78%. For any other atmospheric conditions, a correction factor must be applied. However, since this series of articles deals almost entirely with relative values, the correction factor (which is practically negligible in most cases) will not be considered.

Having determined the ballistic coefficient Co, the next step is to solve the trajectory for a given initial velocity and given angle of departure.

The accurate solution of an actual trajectory is complicated and laborious because the drag is not constant but varies as the square of velocity. The most accurate method is to divide the trajectory into small parts and compute them in sequence.

The next most accurate method is known as the Ingalls-Siacci method which, in making certain assumptions, permits a simpler solution of the problem. No attempt will be made here to describe this method, since a detailed description would fill a book by itself. Suffice it to say that up to very recent years the Ingalls-Siacci method has been used universally for the high degree of accuracy demanded by artillery in the solution of trajectories.

In applying this method, use is made of the Ingalls' Ballistic Tables, by means of which the actual work of computation is materially reduced. These tables are so well known that copies may be found in almost any large public library. In recent years, additional tables have been prepared for velocities below 800 feet per second., where the resistance varies as the square of the velocity , which simplify the work still further. since all our bow and arrow work deals with velocities below 800 feet per second, these additional table have materially reduced the work of solving arrow trajectories.

Having computed the ballistic coefficient Co, formula (8), for a given arrow, the complete trajectory can then be solved for any initial velocity and any angle of departure, by use of these tables. Factors such as range, maximum ordinate, time of flight and striking velocity can all be readily and quickly determined. No further description of these tables will be given here, since complete instructions for their use accompany the tables.

Since in this article we are interested mainly in the maximum flight ranges for a given set of conditions, we can reduce the number of variables and concentrate on the quick solution of certain factors of trajectory.

The first problem is to determine the angle of departure that will give maximum range. By assuming a certain ballistic coefficient Co, and various values of initial velocity so as to give various ranges, the maximum ranges were computed for various angles of departures by use of the tables. The result are plotted in Chart No.1. This shows that as the initial velocity increases, the angle of departure for maximum range decreases, but the variation for quite a number of degrees is so small that by using an average angle of 42 degrees for all conditions , the maximum possible error is a fractional part of 1%.

If we assume standard atmospheric conditions and an angle of departure of 42 degrees, the ranges for a large variation of initial velocities and ballistic coefficients can be compared. The results of such computations are shown plotted in Chart No. 2 and No. 3. Chart No. 2 shows the maximum range plotted as a function of the ballistic coefficient Co for every 40 feet per second velocity between 100 and 500. Chart No. 3 shows the maximum range plotted as a function of velocity for various values of the ballistic coefficient Co.

With these two charts, the maximum flight range for any size and weight arrow , and for any initial velocity between 100 and 500 feet per second can be quickly determined by interpolation. But it cannot give results for the Turkish-shaped, heavy banded arrow. For these, special measurements in wind tunnels seems necessary.

As an example, assume an arrow of 250 grains , .275" diameter, 28" long, and with feathers having a total area including both sides, of 2.0 square inches.

From formula (4) -

$K = 0.0000013 \times 1 \times .275^2 + 0.000000035 \times 28 \times .275 + 0.000000077 \times 2.0 = 0.00000051$

From formula (7) -

$Co = 0.0000000002075 \times 250/0.00000051 = 0.1016$

From Chart No. 2 for an initial velocity of 180 feet per second, for Co = 0.1016, the maximum range is 250 yards.

These ranges represent the maximum range theoretically possible in absolutely still air. Under actual conditions, the resistance of the arrow may be increased due to wind or due to flirting of the arrow in flight, both of which would tend to decrease the range. However, the theoretical consideration is an excellent means for determining the relative effect of various features in bow and arrow design on range, and also indicates the maximum range that can be obtained for a given set of conditions.

Since equation (3) gives the effect of various portions of the arrow on air resistance, we can determine from Chart No. 2 the effects of changes in arrow design on maximum range. The features in arrow design which have no influence on the initial velocity of the arrow shot from a given bow but which affect resistance ad therefore range, are the length, and diameter of the arrow, the shape of the head, and size and type of feather. Weight affects the range by changing the ballistic coefficient Co, but weight also affects the velocity of the arrow from a given bow.

Referring to Chart No. 2, it will be noted that for a given change in the ballistic coefficient Co, the amount of change in range depends upon the value of the coefficient and the initial velocity. The effect of the diameter and the length of arrow, etc., on the range will depend upon the weight and design of the arrow and on the initial velocity. however, in order to obtain some idea as to the effect of these features, we can assume some average condition.

Let us consider the effects of arrow length and diameter. ans size and type of feathers on a flight arrow for ranges between 450 and 500 yards, which would require a bow of 60 to 80 pounds. The weight of the arrow for maximum range under this condition is about 260 grains. The initial velocity would be 300 feet per second.

Assume therefore that the weight of arrow is 260 grains, the initial velocity is 300 feet per second, the shape of the head is ogival, and that the spine remains constant, regardless of the change in length or diameter.

By varying the length and diameter of the arrow, and the size and type of feathers, each in turn , while keeping al the other features constant, we can determine the effect on maximum range.

Co was determined for the various conditions by use of formula (4) and (8), and the ranges were taken from Chart No. 2. The results of these computations are shown in Chart No. 4. The basic arrow was assumed to be 28" long, .275" in diameter, with feathers having a total area, including both sides, of 2.0 square inches. This chart shows the large effect feather size has on flight range, and the increased range to be obtained with celluloid vanes.

CONCLUSIONS

1. An approximate formula (5) has been established for determining the resistance in air of any design arrow. The constants used in the formula were established by wind tunnel tests. Pendulum tests by English check the formula very closely.

2. An accurate and simple method for solving arrow trajectories is available in Ingalls' Ballistic Tables. These tables are used by the Artillery organizations of most large countries, and are sufficiently accurate for all bow and arrow design studies.

3. Use of an angle of departure of 42 degrees will result in only a very small error in determining the maximum range for all bow and arrow conditions.

4. Simple charts (2 and 3) have been developed from Ingalls' Ballistic Table;es for quickly determining the maximum flight ranges of arrows of any design for any initial velocity between 100 and 500 feet per second.

5. The design of an arrow has considerable effect on the maximum range.

6. A change in length of the arrow affects the flight range because of the change in surface area. See Chart No. 4.

7. A change in arrow diameter affects the maximum flight range considerably because of the change in surface area, and also the change in head-on resistance. See Chart No. 4. For example, increasing the arrow diameter from .250" to .300", leaving all other features the same , will decrease the maximum flight range 7%.

8. The size of the feathers has a tremendous effect on maximum range. See Chart No. 4. Increasing the feather area from 2.0 square inches to 6.0 square inches decreases the maximum range 20% or 100 yards out of 500.

9. Celluloid vanes increase the maximum range as compared to feathers. See Chart No. 4. The amount of increase depends upon the size of feather. With a total area of 2.0 square inches for both sides of three feathers, the celluloid vanes increase the maximum flight range 12% or 60 yards out of 500 yards.

10. Further wind tunnel tests on arrows are necessary to establish more accurate coefficients in the formula for air resistance.

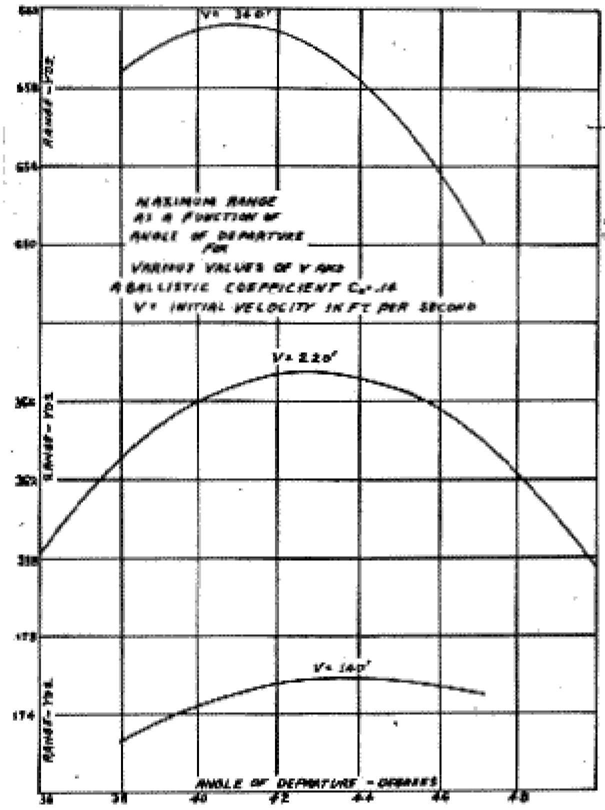
MAXIMUM RANGE
AS A FUNCTION OF
ANGLE OF DEPARTURE
FOR
VARIOUS VALUES OF V AND
A BALLISTIC COEFFICIENT C
V = INITIAL VELOCITY IN FT PER SECOND
V = 220'
V = 140'
RANGE - YDS.
ANGLE OF DEPARTURE - DEGREES
174
38
40
42
44
48

CHART No. 1

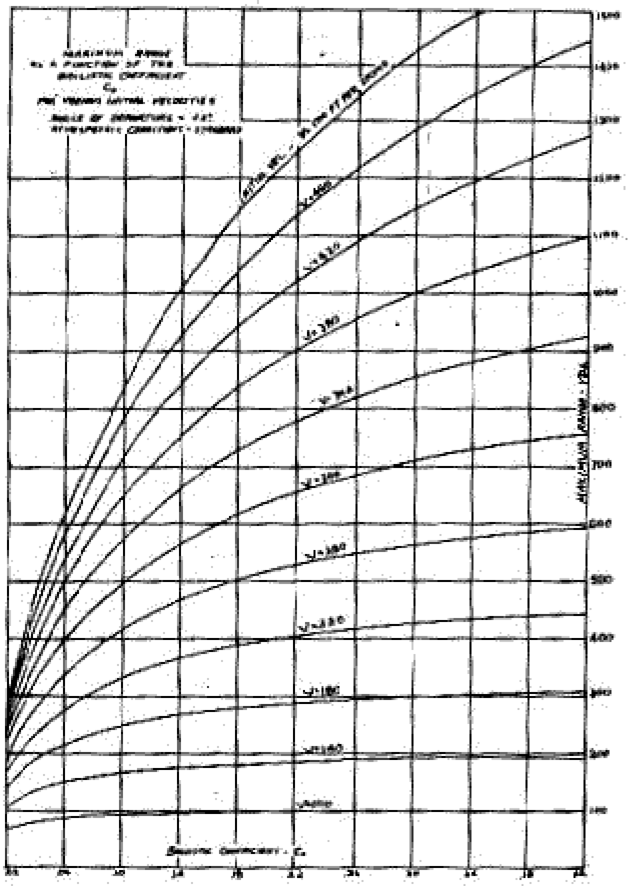

CHART No. 2

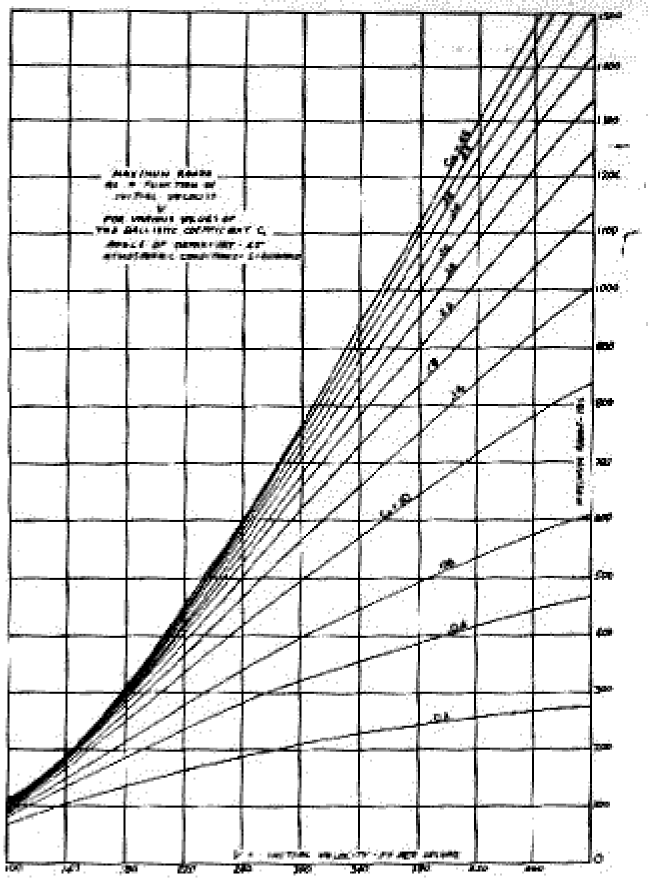

CHART No. 3

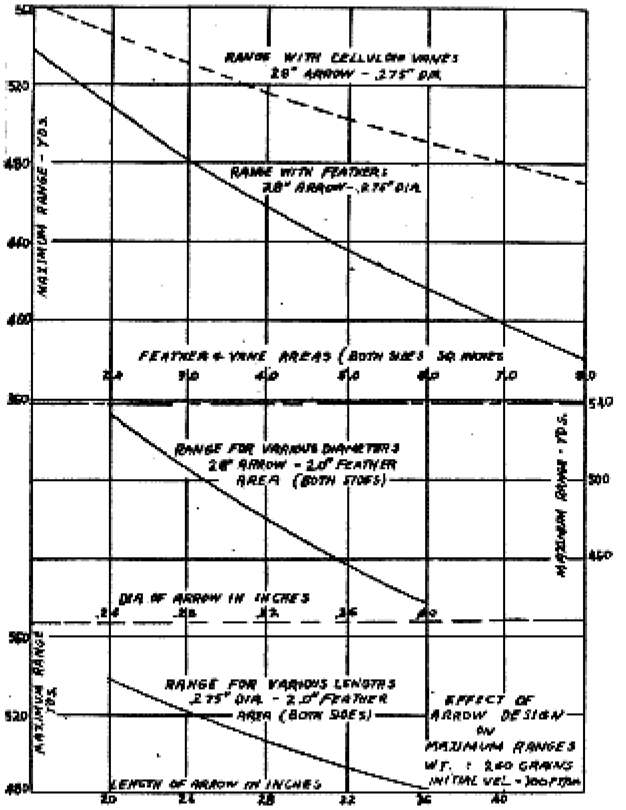

CHART No. 4